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Alternate Spline: A Generalized B-Spline

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A new kind of spline is defined and its properties are studied. It is also proved that the B-splines are actually a special case of this kind of spline. © 1987 Academic Press, Inc.

1. INTRODUCTION

An alternate spline of degree k can be defined in the following way:

DEFINITION 1. Let $\tau = \{\tau_i\}$ be a knot sequence. The ith *alternate spline* of degree k (order $k+1$, $k \geqslant 0$) for the knot sequence τ, denoted by $G_{i,k+1,\tau}$, is defined recursively by the following procedure:

$$G_{i,1,\tau}(x) = \begin{cases} 1, & \tau_i \leqslant x < \tau_{i+1} \\ 0, & \text{otherwise} \end{cases} \tag{1.1}$$

and

$$G_{i,k+1,\tau}(x) = A_{i,k+1,\tau}(x) - A_{i+2,k+1,\tau}(x) \tag{1.2}$$

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for $k \geqslant 1$, where

$$A_{i,k+1,\tau}(x) = \begin{cases} \int_{\tau_i}^{x} G_{i,k,\tau}(s)\, ds/\delta_{i,k,\tau}, & \text{if} \quad \delta_{i,k,\tau} \neq 0 \\ \pi_{i,\tau}(x), & \text{otherwise} \end{cases} \tag{1.3}$$

with

$$\delta_{i,k,\tau} \equiv \int_{\tau_i}^{\tau_{i+2k-1}} G_{i,k,\tau}(s)\, ds \tag{1.4}$$

and

$$\pi_{i,\tau}(x) = \begin{cases} 0, & x < \tau_i \\ 1, & x \geqslant \tau_i \end{cases}. \tag{1.5}$$

Whenever the knot sequence τ can be inferred from the context, we write $G_{i,k}$ instead of $G_{i,k,\tau}$, $A_{i,k}$ instead of $A_{i,k,\tau}$, $\delta_{i,k}$ instead of $\delta_{i,k,\tau}$ and π_i instead of $\pi_{i,\tau}$.

We can use equalities (1.1)–(1.5) to express explicitly the lower degree alternate splines for a given knot sequence. For example, when τ is uniformly spaced, we have

$$G_{i,2}(x) = \begin{cases} (x-\tau_i)/\Delta\tau, & \tau_i \leqslant x < \tau_{i+1} \\ 1, & \tau_{i+1} \leqslant x < \tau_{i+2} \\ (\tau_{i+3}-x)/\Delta\tau, & \tau_{i+2} \leqslant x < \tau_{i+3} \\ 0, & \text{otherwise}, \end{cases}$$

$$G_{i,3}(x) = \begin{cases} (x-\tau_i)^2/4\,(\Delta\tau)^2, & \tau_i \leqslant x < \tau_{i+1} \\ 1/4 + (x-\tau_{i+1})/2\Delta\tau, & \tau_{i+1} \leqslant x < \tau_{i+2} \\ 1-(\tau_{i+3}-x)^2/4(\Delta\tau)^2 & \\ \quad -(x-\tau_{i+2})^2/4(\Delta\tau)^2, & \tau_{i+2} \leqslant x < \tau_{i+3} \\ 3/4-(x-\tau_{i+3})/2\Delta\tau, & \tau_{i+3} \leqslant x < \tau_{i+4} \\ (\tau_{i+5}-x)^2/4(\Delta\tau)^2, & \tau_{i+4} \leqslant x < \tau_{i+5} \\ 0, & \text{otherwise} \end{cases}$$

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and

$$G_{i,4}(x)=\begin{cases} (x-\tau_i)^3/24\,(\Delta\tau)^3, & \tau_i \leqslant x < \tau_{i+1} \\ 1/24+(x-\tau_{i+1})/8\Delta\tau \\ \quad +(x-\tau_{i+1})^2/8\,(\Delta\tau)^2, & \tau_{i+1} \leqslant x < \tau_{i+2} \\ 1/4+(x-\tau_{i+2})/2\Delta\tau \\ \quad +(\tau_{i+3}-x)^3/24(\Delta\tau)^3 \\ \quad -(x-\tau_{i+2})^3/12\,(\Delta\tau)^3, & \tau_{i+2} \leqslant x < \tau_{i+3} \\ 2/3+(x-\tau_{i+3})/4\Delta\tau \\ \quad -(x-\tau_{i+3})^2/4\,(\Delta\tau)^2, & \tau_{i+3} \leqslant x < \tau_{i+4} \\ 3/4-(x-\tau_{i+4})/2\Delta\tau \\ \quad -(\tau_{i+5}-x)^3/12\,(\Delta\tau)^3 \\ \quad +(x-\tau_{i+4})^3/24\,(\Delta\tau)^3, & \tau_{i+4} \leqslant x < \tau_{i+5} \\ 7/24-3(x-\tau_{i+5})/8\Delta\tau \\ \quad +(x-\tau_{i+5})^2/8\,(\Delta\tau)^2, & \tau_{i+5} \leqslant x < \tau_{i+6} \\ (\tau_{i+7}-x)^3/24\,(\Delta\tau)^3, & \tau_{i+6} \leqslant x < \tau_{i+7} \\ 0, & \text{otherwise} \end{cases}$$

where $\Delta\tau$ is the distance between two consecutive knots. Note that $G_{i,2}$ is composed of polynomials of degree one and zero alternately, $G_{i,3}$ is composed of polynomials of degree two and one alternately, and, $G_{i,4}$ is composed of polynomials of degree three and two alternately. Examples of $G_{i,2}$, $G_{i,3}$ and $G_{i,4}$ for a uniformly spaced knot sequence τ are shown in Fig. 1.

When a set of 3D vectors $\{C_i\}$ is given, by using alternate splines defined

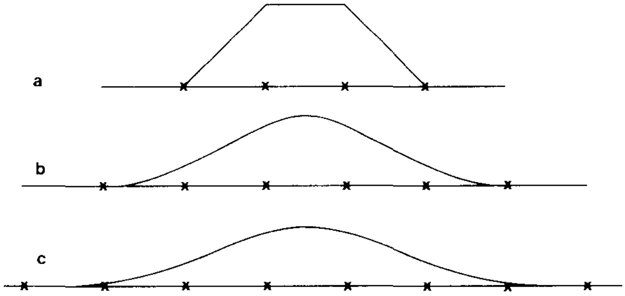

FIG. 1. Examples of alternate splines: (a) $G_{i,2}$, (b) $G_{i,3}$, (c) $G_{i,4}$.

in Definition 1, we can define alternate spline curves of degree k the following way:

$$r(x) = \sum_i C_i G_{2i,k+1,\tau}(x) \tag{1.6}$$

or

$$r(x) = \sum_i C_i G_{2i+1,k+1,\tau}(x). \tag{1.7}$$

Parametric curves constructed this way are of some interest in that they are composed of polynomials of degree k and $k-1$ alternately and, still, are of class C^{k-1} as well be seen later in Section 2.

2. Properties and Proofs

Properties of alternate splines will be discussed in this section. We shall call simple facts "propositions" and leave them without proof.

Let $\tau = \{\tau_l\}$ be a knot sequence and $\tau + x_0 = \{\tau_l + x_0 \mid \tau_l \in \tau\}$, $k \geqslant 0$.

Proposition 1. (i) $G_{i,k+1,\tau+x_0}(x + x_0) = G_{i,k+1,\tau}(x)$,

(ii) $G_{i,k+1,\tau}$ *depends on* $\tau_i, \ldots, \tau_{i+2k+1}$, *only.*

Proposition 2. *The support of* $G_{i,k+1}$, *for all values of* i *and* k, *is finite. More precisely,*

$$G_{i,k+1}(x) = 0 \qquad \textit{for} \quad x \notin [\tau_i, \tau_{i+2k+1}].$$

Proposition 3. *We have*

(i) $$\sum_i G_{2i,k+1,\tau}(x) = \sum_{i=\lceil (r-2k+1)/2 \rceil}^{\lfloor (s-1)/2 \rfloor} G_{2i,k+1,\tau}(x) = 1$$

and

(ii) $$\sum_i G_{2i+1,k+1,\tau}(x) = \sum_{i=\lfloor (r-2k)/2 \rfloor}^{\lceil (s-2)/2 \rceil} G_{2i+1,k+1,\tau}(x) = 1$$

for all $x \in (\tau_r, \tau_s)$.

A principal property of alternate splines is given in the following proposition. We need a definition first.

Definition 2. For the knot sequence $\tau = \{\tau_i\}$, $I_{i+2l} = [\tau_{i+2l}, \tau_{i+2l+1}]$, $l = 0, 1, \ldots, k$, are called the *even intervals* of the alternate spline $G_{i,k+1,\tau}$ and

$I_{i+2l-1}=[\tau_{i+2l-1}, \tau_{i+2l}]$, $l=1, 2,..., k$, are called the *odd intervals* of $G_{i,k+1,\tau}$.

PROPOSITION 4. *$G_{i,k+1,\tau}$ is a polynomial of degree $\leqslant k$ in an even interval and a polynomial of degree $\leqslant k-1$ in an odd interval.*

The next theorem will be discussing the effect of multiple knots in the knot sequence for alternate splines. But, first of all, the definition of multiple knots.

DEFINITION 3. Let $\tau_s=\tau_t$ $(s\leqslant t)$ be two knots contained in $[\tau_i, \tau_{i+2k+1}]$ such that no other knots in $[\tau_i, \tau_{i+2k+1}]$ equal to τ_s except $\tau_{s+1},..., \tau_{t-1}$. If there are n even intervals of $G_{i,k+1,\tau}$, $I_{i+2(j+l)}$, $l=0, 1,..., n-1$, contained in $[\tau_s, \tau_t]$ then $\tau_s=\tau_t$ is called an (n) *multiple knot* of $G_{i,k+1,\tau}$ of multiplicity n.

THEOREM 1. *The following two statements are true for all non-negative integer k:*

I (k): $G_{i,k+1}(x)>0$, $x\in(\tau_i, \tau_{i+2k+1})$

II (k): If τ_s is an (n) multiple knot of $G_{i,k+1}$ and $n\leqslant k-1$ then $G_{i,k+1}$ has continuous $(k-1-n)$th derivative at τ_s but the $(k-n)$th derivative does not exist; if $n=k$ then $G_{i,k+1}$ is not continuous at τ_s; if $n=k+1$ then $G_{i,k+1}=0$.

The proof of this theorem requires several auxiliary results. We will first prove these results and then Theorem 1.

LEMMA 1. *If the statements* I (l) *and* II (l) *in Theorem* 1 *hold for $l=1, 2,..., k-1$ then for any integer j, $0\leqslant j\leqslant k$, we can always find $j+1$ positive real numbers C_l, $l=0, 1,..., j$, such that*

$$G^{(j)}_{i,k+1}(x)=\sum_{l=0}^{j}(-1)^l C_l G_{i+2l,k-j+1}(x) \tag{2.1}$$

for all x in $[\tau_i, \tau_{i+2k+1}]$ except, possibly, at a finite number of knots where (2.1) *does not hold.*

Proof. When $j=0$ the lemma is obviously true. To prove the lemma is true for an arbitrary $j\geqslant 1$, assume the lemma holds for all $m<j$. We have by induction hypothesis that there exist j positive real numbers C_l, $l=0, 1,..., j-1$, such that, except at a finite number of knots,

$$G^{(j-1)}_{i,k+1}(x)=\sum_{l=0}^{j-1}(-1)^l C_l G_{i+2l,k-j+2}(x) \tag{2.2}$$

for all x in $[\tau_i, \tau_{i+2k+1}]$. Now consider $A'_{i+2l,k-j+2}$, $l=0, 1,..., j$, in two different cases: $\delta_{i+2l,k-j+1}=0$ and $\neq 0$.

If $\delta_{i+2l,k-j+1}=0$ then we have by definition that

$$A_{i+2l,k-j+2}=\pi_{i+2l}.$$

Therefore, $A'_{i+2l,k-j+2}$ equals zero at all points except τ_{i+2l}. On the other hand we have by I $(k-j)$ and II $(k-j)$ that

$$G_{i+2l,k-j+1}=0.$$

Hence, in this case, we have

$$A'_{i+2l,k-j+2}(x)=G_{i+2l,k-j+1}(x) \tag{2.3}$$

for all x except the point τ_{i+2l}.

If $\delta_{i+2l,k-j+1}\neq 0$ then $G_{i+2l,k-j+1}$ can have at most one $(k-j)$ multiple knot of multiplicity $k-j$ and it is either τ_{i+2l} or $\tau_{i+2(l+1)}$. Therefore by II $(k-j)$ it can be concluded that $G_{i+2l,k-j+1}$ is continuous everywhere except, possibly, at either τ_{i+2l} or $\tau_{i+2(l+1)}$. But then we have, when $x\in R\backslash\{\tau_{i+2l}, \tau_{i+2(l+1)}\}$,

$$A'_{i+2l,k-j+2}(x)=G_{i+2l,k-j+1}(x)/\delta_{i+2l,k-j+1}. \tag{2.4}$$

Therefore if we set

$$\lambda_{i+2l,k-j+1}=\begin{cases}1/\delta_{i+2l,k-j+1}, & \delta_{i+2l,k-j+1}\neq 0\\ 1, & \text{otherwise,}\end{cases} \tag{2.5}$$

then when $x\in R\backslash\{\tau_{i+2l}, \tau_{i+2(l+1)}\}$ we have, from (2.3) and (2.4),

$$A'_{i+2l,k-j+2}(x)=\lambda_{i+2l,k-j+1}G_{i+2l,k-j+1}(x).$$

This is also true for $A_{i+2(l+1),k-j+2}$. Hence from the definition of $G_{i+2l,k-j+2}$ we have that, for all $x\in R\backslash\{\tau_{i+2l}, \tau_{i+2(l+1)}, \tau_{i+2(l+2)}\}$,

$$\begin{aligned}G'_{i+2l,k-j+2}(x)=\lambda_{i+2l,k-j+1}&G_{i+2l,k-j+1}(x)\\&-\lambda_{i+2(l+1),k-j+1}G_{i+2(l+1),k-j+1}(x),\end{aligned}$$

$l=0, 1,..., j$. Substitute these equations into (2.2) then we have, for all $x\in[\tau_i, \tau_{i+2k+1}]\backslash\{\tau_i, \tau_{i+1},..., \tau_{i+2k+1}\}$, that

$$G^{(j)}_{i,k+1}(x)=\sum_{l=0}^{j}(-1)^l(C_{l-1}+C_l)\,\lambda_{i+2l,k-j+1}G_{i+2l,k-j+1}(x)$$

where $C_{-1}=C_j=0$. From (2.5) and I $(k-j)$ it can be seen that $\lambda_{i+2l,k-j+1}>0$ and the proof of the lemma is complete.

LEMMA 2. *If the statements* I $(k-1)$ *and* II $(k-1)$ *in Theorem* 1 *are true for a positive integer k* $(k \geqslant 1)$ *then*

(i) *If* τ_i *is an* (k) *multiple knot of* $G_{i,k+1}$ *then*

$$G_{i,k+1}(x) = \pi_i(x) - \int_{\tau_{i+2}}^{x} G_{i+2,k}(s)\, ds/\delta_{i+2,k};$$

(ii) *If* τ_{i+2k+1} *is an* (k) *multiple knot of* $G_{i,k+1}$ *then*

$$G_{i,k+1}(x) = \int_{\tau_i}^{x} G_{i,k}(s)\, ds/\delta_{i,k} - \pi_{i+2}(x).$$

Proof. It suffices to prove (i). Since τ_i is a (k) multiple knot of $G_{i,k+1}$, it is also a (k) multiple knot of $G_{i,k}$. We have then by II$(k-1)$ that $G_{i,k}(x) \equiv 0$ or, equivalently, $\delta_{i,k} = 0$. Therefore $A_{i,k+1} = \pi_i$.

Furthermore, since τ_i is a (k) multiple knot of $G_{i,k+1}$, it follows that $\tau_{i+2} \neq \tau_{i+2k+1}$. But then by I $(k-1)$ that

$$\int_{\tau_{i+2}}^{\tau_{i+2k+1}} G_{i+2,k}(s)\, ds \neq 0$$

or, equivalently, $\delta_{i+2,k} \neq 0$. Hence

$$A_{i+2,k+1}(x) = \int_{\tau_{i+2}}^{x} G_{i+2,k}(s)\, ds/\delta_{i+2,k}$$

and (i) follows.

LEMMA 3. *A real-valued function f has n distinct zeros in the interval* $[a, b]$. *If f satisfies the following two conditions:*

(i) *f is continuous at each of these n zeros, and*

(ii) f' *does not exist at m points in* (a, b) *then the number of distinct zeros of* f' *in* (a, b) *is at least* $n-1-m$.

Proof. Assume the n distinct zeros of f are: $a \leqslant x_1 < x_2 < \cdots < x_n \leqslant b$. If f' exists at all the points of the open interval (x_i, x_{i+1}) then by Rolle's Theorem we know that f' has at least one zero in (x_i, x_{i+1}). Since, by (ii), there are at most m distinct open intervals in the $n-1$ open intervals (x_i, x_{i+1}), $i = 1, \ldots, n-1$, which contains one of the points where f' does not exist, therefore, the number of distinct zeros of f' is at least $n-1-m$.

LEMMA 4. *Let* $\{N_1, N_2, \ldots, N_p\}$ $(N_1 < N_2 < \cdots < N_p)$ *be the set of positive integers such that each number in this set is the multiplicity of some* *multiple knot of* $G_{i,k+1}$. $G_{i,k+1}$ *has no multiple knot of multiplicity* k *or greater. For each* $t \in \{1, 2, \ldots, p\}$ *define*

$$A_t = \{\tau_l \mid \tau_l \text{ is a multiple knot of } G_{i,k+1} \text{ with multiplicity} \geqslant N_t\}$$

and set $N_0 = 0$. *Furthermore, for each non-negative integer* j, *let* Z_j *denote the number of distinct zeros of* $G^{(j)}_{i,k+1}$ *in* (τ_i, τ_{i+2k+1}). *Then for any* $t \in \{1, 2, \ldots, p\}$ *if*

$$k-1-N_{p+1-t} \leqslant j < k-1-N_{p-t}$$

and II (k) *in Theorem* 1 *holds then*

$$Z_{j+1} \geqslant Z_j + 1 - |A_{p+1-t}|.$$

Proof. The proof will be discussed in four cases.

I. $\tau_i, \tau_{i+2k+1} \notin A_{p+1-t}$.

Since in this case none of τ_i and τ_{i+2k+1} is a multiple knot of $G_{i,k+1}$ of multiplicity greater than N_{p-t} and $j < k-1-N_{p-t}$, it follows by II (k) that $G^{(j)}_{i,k+1}$ is continuous at τ_i and τ_{i+2k+1}. Hence by Proposition 2 we can conclude that

$$G^{(j)}_{i,k+1}(\tau_i) = G^{(j)}_{i,k+1}(\tau_{i+2k+1}) = 0,$$

i.e., $G^{(j)}_{i,k+1}$ has $Z_j + 2$ distinct zeros in $[\tau_i, \tau_{i+2k+1}]$. Furthermore we can also tell that $G^{(j)}_{i,k+1}$ is continuous at all these zeros because by Proposition 4 and II (k) we know that if $G^{(j)}_{i,k+1}$ exists at x then $G^{(j)}_{i,k+1}$ is continuous there. Next look at the points where $G^{(j+1)}_{i,k+1}$ does not exist. By II (k) we know

$$k-1-N_{p+1-t} \leqslant j < k-1-N_{p-t}$$

then $G^{(j+1)}_{i,k+1}$ does not exist at the points of $A_{p+1-t} \subseteq (\tau_i, \tau_{i+2k+1})$ only. Hence by Lemma 3 we have

$$Z_{j+1} \geqslant (Z_j + 2) - 1 - |A_{p+1-t}|$$

or,

$$Z_{j+1} \geqslant Z_j + 1 - |A_{p+1-t}|.$$

II. $\tau_i \in A_{p+1-t}$, $\tau_{i+2k+1} \notin A_{p+1-t}$.

Since the fact that $j < k-1-N_{p-t}$ implies that $G^{(j)}_{i,k+1}(\tau_{i+2k+1}) = 0$, it follows that $G^{(j)}_{i,k+1}$ has at least $Z_j + 1$ distinct zeros in $[\tau_i, \tau_{i+2k+1}]$ and $G^{(j)}_{i,k+1}$ is continuous at all these points.

Since $G_{i,k+1}^{(j+1)}$ does not exist only at the points of A_{p+1-l} and $\tau_i \in A_{p+1-l}$, it follows that $G_{i,k+1}^{(j+1)}$ does not exist in (τ_i, τ_{i+2k+1}) only at most at $|A_{p+1-l}| - 1$ points. Hence by Lemma 3 we have

$$\begin{aligned} Z_{j+1} &\geqslant (Z_j + 1) - 1 - (|A_{p+1-l}| - 1) \\ &= Z_j + 1 - |A_{p+1-l}|. \end{aligned}$$

III. $\tau_i \notin A_{p+1-l}$, $\tau_{i+2k+1} \in A_{p+1-l}$.

This case can be processed the same way as case II.

IV. $\tau_i, \tau_{i+2k+1} \in A_{p+1-l}$.

$G_{i,k+1}^{(j)}$ has Z_j distinct zeros in (τ_i, τ_{i+2k+1}), and $G_{i,k+1}^{(j)}$ is continuous at these points. Since $G_{i,k+1}^{(j+1)}$ does not exist only at the points of A_{p+1-l} and $\tau_i, \tau_{i+2k+1} \in A_{p+1-l}$, it follows that in (τ_i, τ_{i+2k+1}) $G_{i,k+1}^{(j+1)}$ does not exist at, at most, $|A_{p+1-l}| - 2$ points. Hence by Lemma 3 we have

$$\begin{aligned} Z_{j+1} &\geqslant Z_j - 1 - (|A_{p+1-l}| - 2) \\ &= Z_j + 1 - |A_{p+1-l}| \end{aligned}$$

and the proof of Lemma 4 is complete.

LEMMA 5. *If the statements* I (l), $l = 0, 1, \ldots, k-1$, *and* II (l), $l = 0, 1, \ldots, k$, *are all true then* $G_{i,k+1}(x) \neq 0$ *for all* x *in* (τ_i, τ_{i+2k+1}).

Proof. There are three cases to consider:

I. $G_{i,k+1}$ has $(k+1)$ multiple knot.
In this case (τ_i, τ_{i+2k+1}) is empty and the lemma is obviously true.

II. $G_{i,k+1}$ has (k) multiple knots but no $(k+1)$ multiple knot.
In this case a (k) multiple knot would either be τ_i or τ_{i+2k+1}. Without loss of generality we may assume that τ_i is an (k) multiple knot of $G_{i,k+1}$. In that case we have by Lemma 2 that

$$G_{i,k+1}(x) = \pi_i(x) - \int_{\tau_{i+2}}^{x} G_{i+2,k}(s)\, ds/\delta_{i+2,k}.$$

Then by I $(k-1)$ we have, for $x \in (\tau_i, \tau_{i+2k+1})$, that

$$\delta_{i+2,k} > \int_{\tau_{i+2}}^{x} G_{i+2,k}(s)\, ds > 0$$

or

$$\pi_i(x) > \int_{\tau_{i+2}}^{x} G_{i+2,k}(s)\, ds/\delta_{i+2,k}.$$

Hence

$$G_{i,k+1}(x) = \pi_i(x) - \int_{\tau_{i+2}}^{x} G_{i+2,k}(s)\, ds/\delta_{i+2,k} > 0$$

for all $x \in (\tau_i, \tau_{i+2k+1})$.

III. $G_{i,k+1}$ has no multiple knot of multiplicity k or greater.

In this case let $\{N_1, N_2, \dots, N_p\}$ $(N_1 < N_2 < \cdots < N_p)$ be the set of positive integers such that each number in this set is the multiplicity of some multiple knot of $G_{i,k+1}$. For each $t \in \{1, 2, \dots, p\}$ define

$$A_t = \{\tau_l \mid \tau_l \text{ is a multiple knot of } G_{i,k+1} \text{ with multiplicity} \geqslant N_t\}$$

and set $N_0 = 0$, $N_{p+1} = k-1$. For each nonnegative integer j, let Z_j denote the number of distinct zeros of $G_{i,k+1}^{(j)}$ in (τ_i, τ_{i+2k+1}). Then for each $t \in \{1, 2, \dots, p\}$ by applying Lemma 4 $N_{p+1-t} - N_{p-t}$ times we have

$$Z_{k-1-N_{p-t}} \geqslant Z_{k-1-N_{p+1-t}} + (1 - |A_{p+1-t}|)(N_{p+1-t} - N_{p-t}). \quad (2.6)$$

(2.6) is true even when $t = 0$ as can be seen below.

I. $N_p = N_{p+1} = k-1$.

In this case (2.6) is obviously true when $t = 0$.

II. $N_p < N_{p+1} = k-1$.

Then for any nonnegative integer $j \leqslant k-1-N_p$, $G_{i,k+1}^{(j)}$ is continuous on $[\tau_i, \tau_{i+2k+1}]$ and $G_{i,k+1}^{(j)}(\tau_i) = G_{i,k+1}^{(j)}(\tau_{i+2k+1}) = 0$. Hence $G_{i,k+1}^{(j)}$ has $Z_j + 2$ distinct zeros in $[\tau_i, \tau_{i+2k+1}]$. By applying Rolle's theorem we have, for each $0 \leqslant j < k-1-N_p$,

$$Z_{j+1} \geqslant Z_j + 1.$$

Therefore

$$Z_{k-1-N_p} \geqslant Z_0 + k - 1 - N_p$$

and this is exactly what we have when 0 is substituted into (2.6) for t. Hence (2.6) is true for $t \in \{0, 1, 2, \dots, p\}$.

By adding (2.6)'s up for $t = 0, 1, \dots, p$ we have

$$\begin{aligned} Z_{k-1} &\geqslant Z_0 + \sum_{t=0}^{p} (1 - |A_{p+1-t}|)(N_{p+1-t} - N_{p-t}) \\ &= Z_0 + \sum_{t=1}^{p+1} (1 - |A_t|)(N_t - N_{t-1}). \end{aligned} \quad (2.7)$$

Since

$$\sum_{l=1}^{p+1} (N_l - N_{l-1}) = k-1 \qquad \text{and} \qquad |A_{p+1}|(N_{p+1} - N_p) = 0$$

(2.7) can be further simplified as

$$\begin{aligned} Z_{k-1} &\geq Z_0 + k - 1 - \sum_{l=1}^{p} |A_l|(N_l - N_{l-1}) \\ &= Z_0 + k - 1 - \sum_{l=1}^{m} n_l. \end{aligned} \tag{2.8}$$

Now if $G_{i,k+1}(x) = 0$ for some x in (τ_i, τ_{i+2k+1}), i.e., $Z_0 \geq 1$, then from (2.8) we get

$$Z_{k-1} \geq k - \sum_{l=1}^{m} n_l. \tag{2.9}$$

On the other hand, by Lemma 1 we know there exist k positive numbers C_l, $l = 0, 1, ..., k-1$, such that, except for a finite number of knots where $G_{i,k+1}^{(k+1)}$ does not exist,

$$G_{i,k+1}^{(k+1)}(x) = \sum_{l=0}^{k-1} (-1)^l \, C_l G_{i+2l,2}(x) \tag{2.10}$$

for all x in $[\tau_i, \tau_{i+2k+1}]$. But, if none of τ_i and τ_{i+2k+1} is a multiple knot of $G_{i,k+1}$ then from (2.10) we arrive at the following result:

$$Z_{k-1} = k - 1 - \sum_{l=1}^{m} n_l$$

which is contrary to (2.9). Hence $G_{i,k+1}(x) \neq 0$ for all x in (τ_i, τ_{i+2k+1}).

Now the Proof of Theorem 1. By induction on k. When $k = 0$ the theorem follows directly from (1.1). Now assume the theorem holds for all $m < k$ and prove that it is also true for k. We will prove II (k) first and then I (k). The proof of II (k) is discussed in three cases.

Case I: $n = k + 1$.

In this case both $G_{i,k}$ and $G_{i+2,k}$ have a multiple knot of multiplicity k, and so by II ($k-1$), $G_{i,k} \equiv 0$ and $G_{i+2,k} \equiv 0$. But then $G_{i,k+1} \equiv 0$ too!

Case II: $n=k$.

In this case either τ_i or τ_{i+2k+1} is a (k) multiple knot of $G_{i,k+1}$, say, τ_i. Then by Lemma 2 we have

$$G_{i,k+1}(x)=\pi_i(x)-\int_{\tau_{i+2}}^{x} G_{i+2,k}(s)\,ds/\delta_{i+2,k},$$

and so $G_{i,k+1}$ is not continuous at τ_i.

Case III: $n\leqslant k-1$.

In this case a (n) multiple knot of $G_{i,k}$ will be a (n) or $(n-1)$ multiple knot of $G_{i,k}$. This is also true for $G_{i+2,k}$. No matter which case happens since in this case $(\tau_i, \tau_{i+2k-1})\neq\varnothing$ and $(\tau_{i+2}, \tau_{i+2k+1})\neq\varnothing$ it follows from I $(k-1)$ and the definition of $G_{i,k+1}$ that

$$G_{i,k+1}(x)=\int_{\tau_i}^{x} G_{i,k}(s)\,ds/\delta_{i,k}-\int_{\tau_{i+2}}^{x} G_{i+2,k}(s)\,ds/\delta_{i+2,k}, \tag{2.11}$$

therefore $G_{i,k+1}$ is continuous everywhere. If $n<k-1$ then, since by II $(k-1)$ we know that $G_{i,k}$ and $G_{i+2,k}$ have at least continuous $(k-2-n)$th derivative at (n) multiple knots of $G_{i,k+1}$, it follows from (2.11) that $G_{i,k+1}$ has continuous $(k-1-n)$th derivative at (n) multiple knots. Next we shall show that $G_{i,k+1}^{(k-n)}$ does not exist at (n) multiple knot.

Let τ_{i+2j} be a (n) multiple knot of $G_{i,k+1}$. If $n=k-1$ then $\tau_{i+2j}=\tau_i$, τ_{i+2}, or τ_{i+2k+1}. Say, $\tau_{i+2j}=\tau_i$. Since (τ_i, τ_{i+2k-1}) and $(\tau_{i+2}, \tau_{i+2k+1})\neq\varnothing$, by I $(k-1)$, (1.2) and (1.3) $G_{i,k+1}$ can be expressed as

$$G_{i,k+1}(x)=\int_{\tau_i}^{x} G_{i,k}(s)\,ds/\delta_{i,k}-\int_{\tau_{i+2}}^{x} G_{i+2,k}(s)\,ds/\delta_{i+2,k}.$$

However, since τ_i is a $(k-1)$ multiple knot of $G_{i,k}$, it follows by Lemma 2 that

$$G_{i,k}(s)=\pi_i(s)-\int_{\tau_{i+s}}^{s} G_{i+2,k-1}(t)\,dt/\delta_{i+2,k-1}$$

and so $G_{i,k}$ is not continuous at τ_i. Therefore, the derivative of $G_{i,k+1}(x)$ does not exist at τ_i. The cases when $\tau_{i+2j}=\tau_{i+2}$ and τ_{i+2k+1} can be proved in a similar way.

If $n<k-1$ then by the fact that $G_{i,k+1}^{(k-1-n)}$ is continuous at τ_{i+2j} and Lemma 1 we can find $k-n$ positive numbers C_l, $l=0, 1,..., k-1-n$, and a neighborhood, $N(\tau_{i+2j})$, of τ_{i+2j} such that for all x in $N(\tau_{i+2j})$

$$G^{(k-1-n)}_{i,k+1}(x) = \sum_{l=0}^{k-1-n} (-1)^l C_l G_{i+2l,n+2}(x)$$
$$= (-1)^{j-2} (C_{j-2} G_{i+2(j-2),n+2}(x)$$
$$- C_{j-1} G_{i+2(j-1),n+2}(x) + C_j G_{i+2j,n+2}(x))$$
$$+ \sum_{l \neq j-2,j-1,j} (-1)^l C_l G_{i+2l,n+2}(x). \tag{2.12}$$

Since, by II $(n+1)$, the derivative of the last term of (2.12) exists at τ_{i+2j}, to prove that the derivative of $G_{i,k+1}$ at τ_{i+2j} does not exist, we only have to show that the derivative of

$$C_{j-2} G_{i+2(j-2),n+2} - C_{j-1} G_{i+2(j-1),n+2} + C_j G_{i+2j,n+2} \tag{2.13}$$

at τ_{i+2j} does not exist. Rewrite (2.13) as

$$(C_{j-2} A_{i+2(j-2),n+2} - C_j A_{i+2(j+1),n+2})$$
$$-((C_{j-2} + C_{j-1}) A_{i+2(j-1),n+2} - (C_{j-1} + C_j) A_{i+2j,n+2}). \tag{2.14}$$

Then the first part can be ignored again because derivative of it at τ_{i+2j} exists. Now since τ_{i+2j} is a (n) multiple knot of $G_{i+2j,n+1}$ and a (n) multiple knot of $G_{i+2(j-1),n+1}$, by Lemma 2 the second part of (2.14) can be formed as

$$-\left(a\int^x \tau_{i+2(j-1)} \left(\int^s \tau_{i+2(j-1)} G_{i+2(j-1),n}(t)\,dt\right) ds\right.$$
$$\left.+b\int^x \tau_{i+2j} \left(\int^s \tau_{i+2(j+1)} G_{i+2(j+1),n}(t)\,dt\right) ds\right)$$
$$+\int^x \tau_{i+2(j-1)} (c+d)\, \pi_{i+2j}(s)\,ds \tag{2.15}$$

where

$$a = (C_{j-2} + C_{j-1})/(\delta_{i+2(j-1),n} \cdot \delta_{i+2(j-1),n+1}),$$
$$b = (C_{j-1} + C_j)/(\delta_{i+2(j+1),n} \cdot \delta_{i+2j,n+1}),$$
$$c = (C_{j-2} + C_{j-1})/\delta_{i+2(j-1),n+1} > 0,$$
$$d = (C_{j-1} + C_j)/\delta_{i+2j,n+1} > 0.$$

Derivative of the first part of (2.15) at τ_{i+2j} exists. But derivative of the second part at τ_{i+2j} does not exist. Therefore $G^{(k-n)}_{i,k+1}$ does not exist at the (n) multiple knot τ_{i+2j}.

Now the proof of $I(k)$. If $x \in (\tau_i, \tau_{i+2})$ then by $I(k-1)$ and Proposition 2 we have $A_{i,k+1}(x) > 0$ and $A_{i+2,k+1}(x) = 0$. Therefore

$$G_{i,k+1}(x) > 0 \qquad \text{if} \quad x \in (\tau_i, \tau_{i+2}). \tag{2.16}$$

This is also true for $(\tau_{i+2k-1}, \tau_{i+2k+1})$. Hence to prove I (k) we only have to show that $G_{i,k+1} > 0$ on $[\tau_{i+2}, \tau_{i+2k-1}]$.

Now assume, on the contrary, that $G_{i,k+1}(y) < 0$ for some y in $[\tau_{i+2}, \tau_{i+2k-1}]$. Since $(\tau_i, \tau_{i+2k+1}) \neq \emptyset$, τ_i can not be a $(k+1)$ multiple knot of $G_{i,k+1}$. Hence we have only two cases to consider: τ_i is not a multiple knot, and, τ_i is an (n) multiple knot of $G_{i,k+1}$ but $0 < n \leqslant k$.

Case I. τ_i is not a multiple knot of $G_{i,k+1}$.

In this case $G_{i,k+1}$ must be continuous. For, otherwise, $G_{i,k+1}$ would have a (k) multiple knot and it could only be τ_{i+2k+1}, but then $(\tau_i, \tau_{i+2k+1}) = (\tau_i, \tau_{i+2})$ and by (2.16) we have $G_{i,k+1}(y) > 0$ contrary to the assumption. However, if $G_{i,k+1}$ is continuous in $[\tau_i, \tau_{i+2k+1}]$ and $(\tau_i, \tau_{i+2}) \neq \emptyset$ then by (2.16) and Bolzano's theorem, $G_{i,k+1}$ has a zero in (τ_i, y), a contradiction to Lemma 5.

Case II. τ_i is an (n) multiple knot of $G_{i,k+1}$ and $1 \leqslant n \leqslant k$.

In this case by Lemma 1 there exist $C_l > 0$, $l = 0, 1, \ldots, k-n$, such that

$$G_{i,k+1}^{(k-n)}(x) = \sum_{l=0}^{k-n} (-1)^l C_l G_{i+2l,n+1}(x) \tag{2.17}$$

for all x in $[\tau_i, \tau_{i+2k+1}]$ except at a finite number of knots. Since τ_i is an (n) multiple knot of $G_{i,k+1}$ we have by Lemma 2 that if $n \neq 0$ then

$$G_{i,n+1}(x) = \pi_i(x) - \int_{\tau_{i+2}}^{x} G_{i+2,n}(s)\, ds/\delta_{i+2,n},$$

and consequently

$$G_{i,n+1}(\tau_i) > 0 \qquad \text{and} \qquad G_{i,n+1} \in C(\tau_i, +\infty). \tag{2.18}$$

(2.18) is also true when $n = 0$ by checking the definition of $G_{i,1}$. Furthermore, since τ_i is an (n) multiple knot of $G_{i,k+1}$, it follows that $\tau_{i+2n} < \tau_{i+2n+1}$, and therefore

$$G_{i+2l,n+1} \in C(-\infty, \tau_{i+2n+1}), \qquad l = 1, 2, \ldots, k-n.$$

Consequently, by Proposition 2,

$$G_{i+2l,n+1}(\tau_i) = 0, \qquad l = 1, 2, \ldots, k-n. \tag{2.19}$$

But then by (2.17), (2.18), and (2.19) we have

$$\sum_{l=0}^{k-n} (-1)^l C_l G_{i+2l,n+1}(\tau_i) > 0$$

and

$$\sum_{l=0}^{k-n} (-1)^l C_l G_{i+2l,n+1} \in C[\tau_i, \tau_{i+2n+1}).$$

Therefore there exists an $\varepsilon > 0$ such that

$$G_{i,k+1}^{(k-n)} \in C(\tau_i, \tau_i + \varepsilon) \qquad \text{and} \qquad G_{i,k+1}^{(k-n)}(x) > 0$$
$$\text{for} \qquad x \in (\tau_i, \tau_i + \varepsilon).$$

Since for $j = 1, 2, \ldots, k-n$ we have

$$G_{i,k+1}^{(j-1)}(x) = \int_{\tau_i}^{x} G_{i,k+1}^{(j)}(s)\, ds, \qquad x \in (\tau_i, \tau_i + \varepsilon),$$

it follows that

$$G_{i,k+1}(x) > 0 \qquad \text{for} \qquad x \in (\tau_i, \tau_i + \varepsilon).$$

On the other hand, from the discussion of case I we know that if $G_{i,k+1}$ is not continuous then

$$G_{i,k+1}(x) > 0, \; x \in (\tau_i, \tau_{i+2k+1}) = (\tau_i, \tau_{i+2}).$$

Therefore we only have to consider the case when $G_{i,k+1}$ is continuous. But then by Bolzano's theorem there exists a point $t \in (\tau_i, y)$ such that $G_{i,k+1}(t) = 0$, a contradiction. This completes the proof of Theorem 1.

Corollary 1.1. *The degree of smoothness of* $G_{i,k+1}$ *at* τ_{i+2l-1} $(1 \leqslant l \leqslant k)$ will not be affected if the odd interval I_{i+2l-1} is empty, i.e., $\tau_{i+2l-1} = \tau_{i+2l}$.

In other words, the degrees of smoothness of $G_{i,k+1}$ *at* τ_{i+2l-1} *and* τ_{i+2l} *are the same no matter* τ_{i+2l-1} *equals* τ_{i+2l} *or not.*

Corollary 1.2. $G_{i,k+1}$ *is a polynomial of degree k on even intervals and of degree* $k-1$ *on odd intervals.*

Proof. By taking $j = k-1$ in Lemma 1 and then using Proposition 4.

3. Alternate Splines and B-Splines

In this section we shall prove that B-spline is actually a special case of alternate spline if the knots are chosen properly.

THEOREM 2. *If the knot sequences $\tau = \{\tau_l\}$ and $t = \{t_l\}$ satisfy the conditions $t_l = \tau_{2l} = \tau_{2l-1}$ then for all values of i and k ($\geqslant 1$) we have*

$$G_{2i,k+1,\tau} = G_{2i-1,k+2,\tau} = B_{i,k+1,t}$$

where $B_{i,k+1,t}$ is the ith *B-spline of degree k for the knot sequence t.*

Theorem 2 implies that for a given knot sequence $\tau = \{\tau_l\}$ if all the odd intervals of $G_{i,k+1,\tau}$ are empty then $G_{i,k+1,\tau}$ becomes a B-spline of degree k, and if all the even intervals of $G_{i,k+1,\tau}$ are empty then $G_{i,k+1,\tau}$ becomes a B-spline of degree $k-1$. This property of alternate splines shows that for any given knot sequence t a proper knot sequence τ can always be found so that B-splines for t are equal to the corresponding alternate splines for τ. Therefore, a parametric B-spline curve is also a parametric alternate spline curve, i.e., a parametric B-spline curve can always be represented by alternate splines.

Before we give the proof of Theorem 2, let us recall from [1, p, 131] the definition of B-splines. For a given knot sequence $t = \{t_l\}$, B-splines for the knot sequence t can be recursively defined as follows:

$$B_{i,1,t}(x) = \begin{cases} 1, & t_i \leqslant x < t_{i+1} \\ 0, & \text{otherwise} \end{cases}$$

and

$$B_{i,k+1,t}(x) = \frac{x - t_i}{t_{i+k} - t_i} B_{i,k,t}(x) + \frac{t_{i+k+1} - x}{t_{i+k+1} - t_{i+1}} B_{i+1,k,t}(x)$$

for $k \geqslant 1$. It is easy to see that

$$B_{i,2,t}(x) = \begin{cases} \dfrac{x - t_i}{t_{i+1} - t_i}, & x_i \leqslant x < x_{i+1} \\[2ex] \dfrac{t_{i+2} - x}{t_{i+2} - t_{i+1}}, & x_{i+1} \leqslant x < x_{i+2} \\[2ex] 0, & \text{otherwise.} \end{cases}$$

Now the proof of Theorem 2. By induction on k. It suffices to prove that $G_{2i,k+1,\tau} = B_{i,k+1,t}$. Let $k = 1$. By definition 1 we have

$$G_{2i,2,\tau}(x) = \begin{cases} (x-\tau_{2i})/(\tau_{2i+1}-\tau_{2i}), & \tau_{2i} \leqslant x < \tau_{2i+1} \\ 1, & \tau_{2i+1} \leqslant x < \tau_{2i+2} \\ (\tau_{2i+3}-x)/(\tau_{2i+3}-\tau_{2i+2}), & \tau_{2i+1} \leqslant x < \tau_{2i+3} \\ 0, & \text{otherwise.} \end{cases}$$

Since $\tau_{2i+1} = \tau_{2i+2}$, i.e., $[\tau_{2i+1}, \tau_{2i+2}) = \varnothing$, and $t_i = \tau_{2i}$, $t_{i+1} = \tau_{2i+1}$, $t_{i+2} = \tau_{2i+3}$, it follows that

$$G_{2i,2,\tau}(x) = \begin{cases} (x-t_i)/(t_{i+1}-t_i), & t_i \leqslant x < t_{i+1} \\ (t_{i+2}-x)/(t_{i+2}-t_{i+1}), & t_{i+1} \leqslant x < t_{i+2} \\ 0, & \text{otherwise} \end{cases}$$
$$= B_{i,2,t}(t).$$

Now assume $G_{2i,m+1,\tau} = B_{i,m+1,t}$ for all $m < k$. Then by assumption we have

$$A_{2i,k+1,\tau}(x) = \begin{cases} \int_{t_i}^{x} B_{i,k,t}(s)\,ds \Big/ \int_{t_i}^{t_{i+k}} B_{i,k,t}(s)\,ds, & \\ \qquad \text{if } \int_{t_i}^{t_{i+k}} B_{i,k,t}(s)\,ds \neq 0 & \\ \pi_{i,t}(x), \quad \text{otherwise.} & \end{cases} \tag{3.1}$$

From [1, p. 151] with a slight modification we have the following formula for the integration of B-splines

$$\int_{t_i}^{x} \sum_{l=i}^{n} \alpha_l B_{l,k,t}(s)\,ds = \sum_{l=i}^{m} \left(\sum_{j=i}^{l} \alpha_j (t_{j+k} - t_j)/k \right) B_{l,k+1,t}(x),$$
$$x \leqslant t_{m+1}. \tag{3.2}$$

Therefore for any real number x by choosing a sufficiently large m so that $x \leqslant t_{m+1}$ then by (3.2) we have

$$\int_{t_i}^{x} B_{i,k,t}(s)\,ds = (t_{i+k} - t_i) \left(\sum_{l=i}^{m} B_{l,k+1,t}(s) \right) \Big/ k \tag{3.3}$$

and, similarly,

$$\int_{t_i}^{t_{i+k}} B_{i,k,t}(s)\, ds = (t_{i+k} - t_i)/k. \tag{3.4}$$

(3.4) follows from the fact that

$$\sum_i B_{i,k,t}(x) = \sum_{i=r+1-k}^{s-1} B_{i,k,t}(x) = 1 \tag{3.5}$$

if $t_r \leqslant x \leqslant t_s$ [1, p. 110]. Hence, by (3.1), (3.3), and (3.4),

$$A_{2i,k+1,\tau}(x) = \sum_{l=i}^{m} B_{l,k+1,t}(x) \qquad \text{if} \quad \int_{t_i}^{t_{i+k}} B_{i,k,t}(s)\, ds \neq 0.$$

If

$$\int_{t_i}^{t_{i+k}} B_{i,k,t}(s)\, ds = 0, \qquad \text{i.e.,} \quad \int_{\tau_{2i}}^{\tau_{2i+2k-1}} G_{2i,k,\tau}(s)\, ds = 0,$$

then, by Theorem 1, $\tau_{2i} = \tau_{2i+2k-1}$, or, $t_i = t_{i+k}$ and therefore by (3.5) we also have

$$\begin{aligned} A_{2i,k+1,\tau}(x) &= \pi_{i,t}(x) \\ &= \sum_{l=i}^{m} B_{l,k+1,t}(x) \qquad \text{if} \quad \int_{t_i}^{t_{i+k}} B_{i,k,t}(s)\, ds = 0. \end{aligned}$$

Therefore for any real number x, by choosing m large enough so that $x \leqslant t_{m+1}$, we always have the following equation

$$A_{2i,k+1,\tau}(x) = \sum_{l=i}^{m} B_{l,k+1,t}(x). \tag{3.6}$$

Similarly we can prove that, for any real number x, by choosing m large enough so that $x \leqslant t_{m+1}$ then

$$A_{2i+2,k+1,\tau}(x) = \sum_{l=i+1}^{m} B_{l,k+1,t}(x) \tag{3.7}$$

(remember that $\tau_{2i+1} = \tau_{2i+2}$), and the theorem follows from (3.6), (3.7), and (1.2).

Representation of alternate splines by B-splines is given in the following theorem:

THEOREM 3. *If $G_{i,k+1,\tau}$ is the* ith *alternate spline of degree k ($\geqslant 0$) for* *the knot sequence* $\tau = \{\tau_l\}$ *then there exists* $k+1$ *real numbers* $\alpha^{(k+1)}_{l,i}$, $l = 0, 1, \ldots, k$, *such that*

$$G_{i,k+1,\tau}(x) = \sum_{l=0}^{k} \alpha^{(k+1)}_{l,i} B_{i+l,k+1,\tau}(x)$$

for all x where $\alpha^{(k+1)}_{l,i}$, $l = 0, 1, \ldots, k$, can be defined recursively for k as follows:

$$\alpha^{(1)}_{0,i} = 1,$$

and

$$\alpha^{(k+1)}_{l,i} = a^{(k+1)}_{l,i} - a^{(k+1)}_{l-2,i+2}, \qquad l = 0, 1, \ldots, k$$

for $k > 0$ with

$$a^{(k+1)}_{l,i} = \begin{cases} \sum_{j=i}^{i+l} \alpha^{(k)}_{j-i,i}(\tau_{j+k} - \tau_j)/\Delta^{(k)}_i, & \text{if } \Delta^{(k)}_i \neq 0 \\ 1, & \text{otherwise,} \end{cases}$$

$$\Delta^{(k)}_i = \sum_{j=i}^{i+k-1} \alpha^{(k)}_{j-i,i}(\tau_{j+k} - \tau_j), \tag{3.8}$$

and

$$\alpha^{(k)}_{k,i} = a^{(k+1)}_{-2,i+2} = a^{(k+1)}_{-1,i+2} = 0.$$

Proof. By induction on k. When $k = 0$ the theorem follows directly from the definition of $G_{i,1,\tau}$ and $B_{i,1,\tau}$. Now assume the theorem holds for all $m < k$. We have by induction hypothesis that

$$G_{i,k,\tau}(s) = \sum_{l=0}^{k-1} \alpha^{(k)}_{l,i} B_{i+l,k,\tau}(s).$$

By finding m large enough so that $x \leqslant \tau_{m+1}$ then by (3.2) we have

$$\int_{\tau_i}^{x} G_{i,k,\tau}(s)\, ds = \sum_{l=i}^{m} \left(\sum_{j=i}^{l} \alpha^{(k)}_{j-i,i} (\tau_{j+k} - \tau_j)/k \right) B_{l,k+1,\tau}(x). \tag{3.9}$$

Since $B_{i,k+1,\tau}(t) = 0$ if $t \notin [\tau_i, \tau_{i+k+1}]$, it follows that

$$\int_{\tau_i}^{\tau_{i+2k-1}} G_{i,k,\tau}(s)\, ds$$

$$= \sum_{l=i+k-1}^{m} \left(\sum_{j=i}^{l} \alpha^{(k)}_{j-i,i}(\tau_{j+k} - \tau_j)/k \right) B_{l,k+1,\tau}(\tau_{i+2k-1}).$$

Furthermore, since $\alpha_{l,i}^{(k)}=0$ if $l>k-1$, we have

$$\begin{aligned}\int_{\tau_i}^{\tau_{i+2k-1}} G_{i,k,\tau}(s)\,ds &= \sum_{l=i+k-1}^{m}\left(\sum_{j=i}^{i+k-1}\alpha_{j-l,i}^{(k)}(\tau_{j+k}-\tau_j)/k\right)B_{l,k+1,\tau}(\tau_{i+2k-1})\\ &= \sum_{l=i+k-1}^{m}\Delta_i^{(k)}B_{l,k+1,\tau}(\tau_{i+2k-1})/k\end{aligned}$$

and then by (3.5)

$$\int_{\tau_i}^{\tau_{i+2k-1}} G_{i,k,\tau}(s)\,ds=\Delta_i^{(k)}/k$$

or

$$\delta_{i,k,\tau}=\Delta_i^{(k)}/k. \tag{3.10}$$

Again, since $\alpha_{l,i}^{(k)}=0$ if $l>k-1$, (3.9) can be simplified as

$$\begin{aligned}\int_{\tau_i}^{x} G_{i,k,\tau}(s)\,ds = &\sum_{l=i}^{i+k-2}\left(\sum_{j=i}^{l}\alpha_{j-l,i}^{(k)}(\tau_{j+k}-\tau_j)/k\right)B_{l,k+1,\tau}(x)\\ &+\sum_{l=i+k-1}^{m}(\Delta_i^{(k)}/k)\,B_{l,k+1,\tau}(x).\end{aligned}$$

Therefore if $\delta_{i,k,\tau}\neq 0$ then by (1.3) and (3.10)

$$\begin{aligned}A_{i,k+1,\tau}(x)= &\sum_{l=i}^{i+k-2}\left(\sum_{j=i}^{l}\alpha_{j-l,i}^{(k)}(\tau_{j+k}-\tau_j)/\Delta_i^{(k)}\right)B_{l,k+1,\tau}(x)\\ &+\sum_{l=i+k-1}^{m}B_{l,k+1,\tau}(x)\end{aligned}$$

or

$$\begin{aligned}A_{i,k+1,\tau}(x)= &\sum_{l=0}^{k}\left(\sum_{j=i}^{i+l}\alpha_{j-l,i}^{(k)}(\tau_{j+k}-\tau_j)/\Delta_i^{(k)}\right)B_{i+l,k+1,\tau}(x)\\ &+\sum_{l=i+k+1}^{m}B_{l,k+1,\tau}(x).\end{aligned} \tag{3.11}$$

This is because that, from the definition of $\Delta_i^{(k)}$ and induction hypothesis,

$$\sum_{j=i}^{i+l}\alpha_{j-l,i}^{(k)}(\tau_{j+k}-\tau_j)/\Delta_i^{(k)}=1$$

if $l=k-1$ or k. On the other hand, if $\delta_{i,k,\tau}=0$ then by (1.4) and Theorem 1 we can conclude that $\tau_i=\tau_{i+2k-1}$. Hence by (3.5) we have

$$A_{i,k+1,\tau}(x)=\pi_{i,\pi}(x)=\sum_{l=i}^{m} B_{l,k+1,\tau}(x)$$

for a sufficiently large m, or

$$\begin{aligned} A_{i,k+1,\tau}(x) &= \sum_{l=i}^{i+k} B_{l,k+1,\tau}(x)+\sum_{l=i+k+1}^{m} B_{l,k+1,\tau}(x) \\ &= \sum_{l=0}^{k} B_{i+l,k+1,\tau}(x)+\sum_{l=i+k+1}^{m} B_{l,k+1,\tau}(x). \end{aligned} \tag{3.12}$$

Therefore by (3.11) and (3.12) we then get

$$\begin{aligned} A_{i,k+1,\tau}(x) &= \sum_{l=0}^{k} a_{l,i}^{(k+1)} B_{i+l,k+1,\tau}(x) \\ &\quad + \sum_{l=i+k+1}^{m} B_{l,k+1,\tau}(x) \end{aligned} \tag{3.13}$$

with $a_{l,i}^{(k+1)}$ defined as in (3.8). Similarly, it can be proved that

$$\begin{aligned} A_{i+2,k+1,\tau}(x) &= \sum_{l=0}^{k} a_{l,i+2}^{(k+1)} B_{i+2+l,k+1,\tau}(x) \\ &\quad + \sum_{l=i+k+3}^{m} B_{l,k+1,\tau}(x) \end{aligned}$$

for the same m. Since $a_{k-1,i+2}^{(k+1)}=a_{k,i+2}^{(k+1)}=1$ it follows that

$$\begin{aligned} A_{i+2,k+1,\tau}(x) &= \sum_{l=0}^{k-2} a_{l,i+2}^{(k+1)} B_{i+2+l,k+1,\tau}(x)+\sum_{l=i+k+1}^{m} B_{l,k+1,\tau}(x) \\ &= \sum_{l=0}^{k} a_{l-2,i+2}^{(k+1)} B_{i+l,k+1,\tau}(x)+\sum_{l=i+k+1}^{m} B_{l,k+1,\tau}(x) \end{aligned} \tag{3.14}$$

by setting $a_{-2,i+2}^{(k+1)}=a_{-1,i+2}^{(k+1)}=0$. The theorem now follows from (3.13), (3.14), and (1.2).

COROLLARY 3.1. *If the knot sequence $\tau=\{\tau_i\}$ is uniformly spaced then for integers k (>0) and i we have*

$$G_{i,k+1,\tau}(x)=\sum_{l=0}^{k}\binom{k}{l} B_{i+l,k+1,\tau}(x)/2^{k-1}.$$

Proof. $\alpha_{l,1}^{(k+1)}$ is determined by the relative position of $\tau_i, \tau_{i+1}, ..., \tau_{i+2k+1}$. Hence if τ is uniformly spaced then, for any $l \in \{0, 1, ..., k\}$ and integers s and t, we have

$$\alpha_{l,s}^{(k+1)} = \alpha_{l,t}^{(k+1)}.$$

If we replace $\alpha_{l,i}^{(k+1)}$ by $\alpha_l^{(k+1)}$ and simplify the recurrence relation of $\alpha_{l,i}^{(k+1)}$ in Theorem 3 to be as follows:

$$\alpha_0^{(2)} = \alpha_1^{(2)} = 1$$

$$\alpha_l^{(k+1)} = (\alpha_{l-1}^{(k)} + \alpha_l^{(k)}) \Big/ \sum_{j=0}^{k-1} \alpha_j^{(k)}, \qquad l = 0, 1, ..., k$$

where $\alpha_{-1}^{(k)} = \alpha_k^{(k)} = 0$, then it is easy to see by induction that

$$\sum_{j=0}^{k} \alpha_j^{(k+1)} = 2 \quad \text{and} \quad \alpha_l^{(k+1)} = \binom{k}{l} \Big/ 2^{k-1}$$

and the proof of the corollary is complete

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